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The bifurcated harmonic oscillator

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Abstract

Some general properties of a bifurcated oscillator potential in one dimension are analysed. Appropriate wavefunctions in different regions with continuity conditions lead to a simple relation for the energies. The properties of these energies for small and large values of separation are used to develop simple and accurate expressions for the energies of all the states for all values of separation.

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1. Introduction

The simple harmonic oscillator is a topic of great importance. It is of major theoretical interest and has practical applications in many branches of physics [1]. It has many special properties; in particular, one has closed solutions to the Schrödinger equation in terms of Hermite polynomials or confluent hypergeometric functions. Many variations of the harmonic oscillator have been considered [2]: oscillator inside a well [3], double oscillator [4] and others. An important case which deserves special attention is a bifurcated harmonic oscillator for which we define the potential as

$$\begin{aligned} V(x) &= 0 \quad \text{for } |x| < a, \\ &= \frac{1}{2}k(|x| - a)^2 \quad \text{for } |x| \geq a. \end{aligned} \quad (1.1)$$

It is essentially a potential with the two oscillator components separated with zero potential in between, as shown in figure 1 for the special case of $k = 1$, $a = 1$. It brings in several interesting points about the general properties of the system, dependence of the wavefunctions and energies on the separation parameter a , significance of different approximations, and general expressions for the energies which provide a physical insight into the properties of the system. Apart from the theoretical interest, it may be used to simulate the potential experienced by electrons inside a thin semiconductor layer or in a junction [5], with the potential increasing smoothly at the edges.

Here, the bifurcated harmonic oscillator is analysed in detail. The scaling property is considered and the general solutions are obtained with appropriate boundary conditions. The perturbative solutions are considered for small and large values of the separation parameter a . The WKB solutions for the energies are obtained. They provide an insight into the general

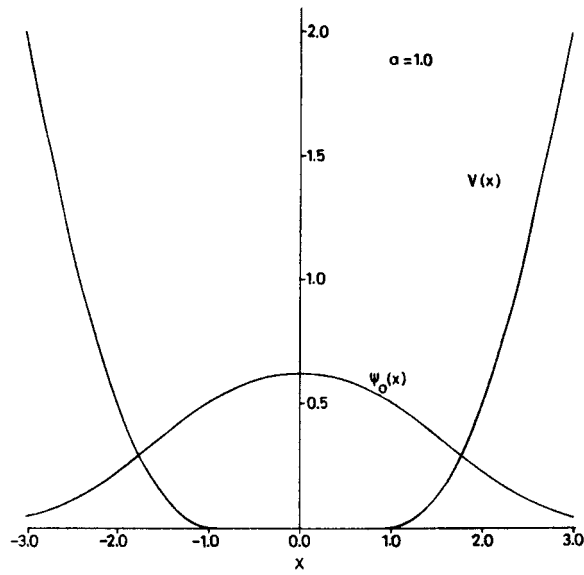


Figure 1. Bifurcated harmonic oscillator potential in equation (2.6) for the separation parameter $a = 1$, and the corresponding ground-state wavefunction.

form of the energies. We then develop simple expressions for the energies of all the states, for all values of separation a . The considerations are extended to the corresponding, spherically symmetric potential in three dimensions.

2. General approach

We analyse the general properties of the bifurcated harmonic oscillator and obtain the energy eigenfunctions and eigenvalues.

2.1. Scaling property

The time-independent Schrödinger equation for the bifurcated harmonic oscillator is

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + \frac{1}{2}k(|x| - a)^2\theta(|x| - a)\psi = E\psi \quad (2.1)$$

where θ is the Heaviside step function, and a may be described as the separation parameter. With the usual scale transformation

$$x = \lambda u \quad (2.2)$$

one obtains

$$-\frac{1}{2} \frac{d^2}{du^2} \psi + \frac{1}{2} \frac{mk}{\hbar^2} \lambda^4 (|u| - a/\lambda)^2 \theta(|u| - a/\lambda) \psi = \frac{m\lambda^2}{\hbar^2} E \psi. \quad (2.3)$$

Taking

$$\lambda = \left(\frac{\hbar^2}{mk} \right)^{1/4}, \quad (2.4)$$

and comparing with equation (2.1) with $m/\hbar^2 = 1, k = 1$, leads to

$$E(m/\hbar^2, k, a) = \hbar(k/m)^{1/2} E(1, 1, a/\lambda), \quad \lambda = \left(\frac{\hbar^2}{mk}\right)^{1/4}, \tag{2.5}$$

$$\psi(m/\hbar^2, k, a, x) = N\psi(1, 1, a/\lambda, x/\lambda).$$

Therefore one needs to consider only the special case,

$$-\frac{1}{2} \frac{d^2\psi}{dx^2} + \frac{1}{2} (|x| - a)^2 \theta(|x| - a) \psi = E\psi, \tag{2.6}$$

with $m/\hbar^2 = 1, k = 1$, and use the scaling relation in equation (2.5) to obtain the solutions for the general case.

2.2. General solutions

For obtaining general solutions, we first note that the solutions can be taken to be odd or even under $x \rightarrow -x$. For $|x| < a$, we have

$$\psi_0 = A_0 \cos\left(px + \eta \frac{\pi}{2}\right), \quad p = (2E)^{1/2}, \quad |x| \leq a, \tag{2.7}$$

where $\eta = 0$ for even solutions and $\eta = 1$ for odd solutions. For writing the solutions for $|x| > a$, since they are taken to be the eigenstates of parity, we will consider the solutions only for $x > a$. In this domain it is convenient to use the variable $y = x - a$, so that

$$-\frac{1}{2} \frac{d^2\psi_+}{dy^2} + \frac{1}{2} y^2 \psi_+ = E\psi_+, \quad y = x - a \geq 0. \tag{2.8}$$

In this region, one can separate out the asymptotic Gaussian part and obtain the solutions in terms of confluent hypergeometric functions,

$$\psi_+(y) = A_+ \left[F\left(\frac{1}{4} - \frac{E}{2}, \frac{1}{2}, y^2\right) + B_+ y F\left(\frac{3}{4} - \frac{E}{2}, \frac{3}{2}, y^2\right) \right] e^{-\frac{1}{2}y^2}, \quad y \geq 0. \tag{2.9}$$

For large values of the variable, the confluent hypergeometric function has the asymptotic behaviour [6, 7]

$$F(a, b, z) \rightarrow \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} e^z, \quad z \rightarrow \infty. \tag{2.10}$$

With this, one has

$$\psi_+(y) \rightarrow A_+ \left[\frac{\Gamma(1/2)}{\Gamma(\frac{1}{4} - \frac{E}{2})} + B_+ \frac{\Gamma(3/2)}{\Gamma(\frac{3}{4} - \frac{E}{2})} \right] y^{-\frac{1}{4} - \frac{E}{2}} e^{\frac{1}{2}y^2} \quad \text{for } y \rightarrow \infty. \tag{2.11}$$

Requirement of the normalizability of the wavefunction implies

$$B_+ = -\frac{\Gamma(1/2)\Gamma(\frac{3}{4} - \frac{E}{2})}{\Gamma(3/2)\Gamma(\frac{1}{4} - \frac{E}{2})}$$

$$= -\frac{2\Gamma(\frac{3}{4} + \frac{E}{2})\sin[\pi(\frac{1}{4} - \frac{E}{2})]}{\Gamma(\frac{1}{4} + \frac{E}{2})\sin[\pi(\frac{3}{4} - \frac{E}{2})]}, \tag{2.12}$$

and therefore

$$\psi_+(y) = A_+ \left[F\left(\frac{1}{4} - \frac{E}{2}, \frac{1}{2}, y^2\right) - \frac{2\Gamma(\frac{3}{4} + \frac{E}{2})\tan(\pi(\frac{1}{4} - \frac{E}{2}))}{\Gamma(\frac{1}{4} + \frac{E}{2})} y F\left(\frac{3}{4} - \frac{E}{2}, \frac{3}{2}, y^2\right) \right],$$

for $y \geq 0$. (2.13)

Table 1. The accurate values of energies obtained from equation (2.15), for the $N = 0, \dots, 5$ states, for some values of the separation parameter a . The values in the brackets are the corresponding values from the simple expression in equation (4.7) for the ground state, and equation (4.13) for the excited states.

a	E_0	E_1	E_2	E_3	E_4	E_5
0.01	0.4944 (0.4944)	1.4888 (1.4890)	2.4859 (2.4859)	3.4831 (3.4832)	4.4810 (4.4809)	5.4789 (5.4789)
0.10	0.4475 (0.4477)	1.3910 (1.3937)	2.3629 (2.3616)	3.3348 (3.3356)	4.3136 (4.3130)	5.2925 (5.2929)
0.20	0.4022 (0.4026)	1.2898 (1.2951)	2.2336 (2.2310)	3.1775 (3.1790)	4.1349 (4.1339)	5.0929 (5.0936)
0.50	0.2988 (0.3002)	1.0296 (1.0408)	1.8863 (1.8824)	2.7515 (2.7531)	3.6416 (3.6410)	4.5407 (4.5408)
1.0	0.1954 (0.1973)	0.7204 (0.7314)	1.4238 (1.4252)	2.1754 (2.1729)	2.9512 (2.9529)	3.7561 (3.7553)
1.5	0.1368 (0.1387)	0.5218 (0.5282)	1.0827 (1.0909)	1.7283 (1.7260)	2.4061 (2.4055)	3.1148 (3.1161)
2.0	0.1009 (0.1025)	0.3917 (0.3947)	0.8365 (0.8479)	1.3810 (1.3834)	1.9744 (1.9718)	2.5973 (2.5978)
3.0	0.061 20 (0.062 25)	0.2414 (0.2417)	0.5300 (0.5404)	0.9089 (0.9196)	1.3537 (1.3571)	1.8404 (1.8388)
5.0	0.029 34 (0.029 81)	0.1168 (0.1165)	0.2608 (0.2654)	0.4583 (0.4678)	0.7051 (0.7162)	0.9951 (1.0039)
10.0	0.009 36 (0.009 47)	0.0374 (0.0373)	0.0841 (0.0848)	0.1491 (0.1513)	0.2324 (0.2364)	0.3336 (0.3394)

The corresponding solutions for $-2a \leq y \leq 0$, given in equation (2.7), are

$$\psi_0(y) = A_0 \cos\left(py + pa - \eta \frac{\pi}{2}\right), \quad p = (2E)^{1/2}, \quad -2a \leq y \leq 0. \quad (2.14)$$

The continuity of the wavefunction and its derivative at $y = 0$ lead to

$$p \tan\left(pa - \eta \frac{\pi}{2}\right) = 2 \frac{\Gamma\left(\frac{3}{4} + \frac{E}{2}\right)}{\Gamma\left(\frac{1}{4} + \frac{E}{2}\right)} \tan\left[\pi \left(\frac{1}{4} - \frac{E}{2}\right)\right], \quad p = (2E)^{1/2}. \quad (2.15)$$

This equation allows us to obtain the energy eigenvalues. The calculated values of the energies of the first six states obtained from equation (2.15), for some values of a are given in table 1. To provide a qualitative understanding of the changes introduced by the variation of the separation parameter a , we have plotted the ground-state wavefunction in figure 1, and the energies as functions of a in figure 2.

3. Some approximations

Here we consider some approximations which provide useful insight and help in developing simple expressions for the energies.

3.1. Energies for small values of a

To begin with, one observes that for $a = 0$, equation (2.15) leads to

$$\begin{aligned} \tan\left[\pi \left(\frac{1}{4} - \frac{E}{2}\right)\right] &= 0 \quad \text{for } \eta = 0, \quad \Rightarrow \quad E = 2n + \frac{1}{2} \\ &= \infty \quad \text{for } \eta = 1, \quad \Rightarrow \quad E = 2n + \frac{3}{2}, \end{aligned} \quad (3.1)$$

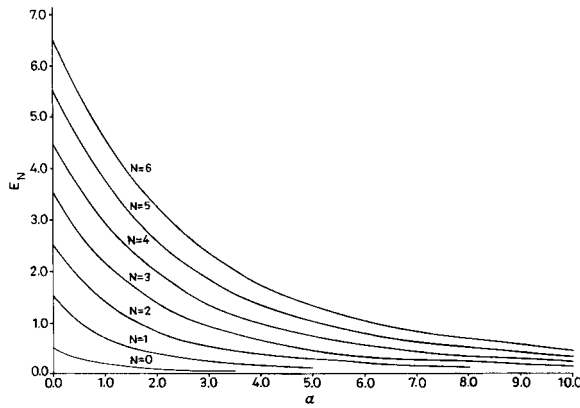


Figure 2. The energies of the first six eigenstates for the bifurcated harmonic oscillator potential in equation (2.6), as functions of the separation parameter a .

so that the energy eigenvalues are

$$E = 2n + \eta + \frac{1}{2} \tag{3.2}$$

which are the appropriate harmonic oscillator energies.

For small values of a , we take

$$E = 2n + \eta + \frac{1}{2} + \delta, \tag{3.3}$$

where δ is small. Then equation (2.15) leads to, for $\eta = 0$,

$$\begin{aligned} 2Ea &= -2 \frac{\Gamma(n+1)}{\Gamma(n+1/2)} \left(\frac{\pi\delta}{2}\right) \\ \Rightarrow \delta &= -\frac{2}{\pi} (2n+1/2) \frac{\Gamma(n+1/2)}{n!} a, \quad \eta = 0, \end{aligned} \tag{3.4}$$

and for $\eta = 1$,

$$\begin{aligned} -\frac{1}{a} &= 2 \frac{\Gamma(n+1+1/2)}{\Gamma(n+1)} \left(\frac{2}{\pi\delta}\right) \\ \Rightarrow \delta &= -\frac{4}{\pi} \frac{\Gamma(n+3/2)}{n!} a, \quad \eta = 1, \end{aligned} \tag{3.5}$$

which together imply

$$\delta = -\frac{2}{\pi} \left(2n + \frac{1}{2}\eta + \frac{1}{2}\right) \frac{\Gamma(n+1/2)}{n!} a. \tag{3.6}$$

Therefore, the energies correct to first order in the separation parameter a , are

$$E = \left(2n + \eta + \frac{1}{2}\right) - \frac{2}{\pi} \left(2n + \frac{1}{2}\eta + \frac{1}{2}\right) \frac{\Gamma(n+1/2)}{n!} a, \quad \text{for small } a. \tag{3.7}$$

3.2. Feynman–Hellmann theorem

We consider the implications of the Feynman–Hellmann theorem by regarding a as a parameter,

$$\left. \frac{\partial E}{\partial a} \right|_{a=0} = \langle \psi | \left. \frac{\partial V}{\partial a} \right|_{a=0} | \psi \rangle. \tag{3.8}$$

With our potential in equation (2.1), this leads to

$$\left. \frac{\partial E}{\partial a} \right|_{a=0} = -\langle \psi ||x| |\psi \rangle |_{a=0}, \quad (3.9)$$

so that

$$E = \left(2n + \eta + \frac{1}{2}\right) - \langle \psi ||x| |\psi \rangle |_{a=0} a, \quad \text{for small } a. \quad (3.10)$$

Comparing this with the result in equation (3.7), we get

$$\langle \psi ||x| |\psi \rangle = \frac{2}{\pi} \left(2n + \frac{1}{2}\eta + \frac{1}{2}\right) \frac{\Gamma(n + 1/2)}{n!}, \quad (3.11)$$

for the harmonic oscillator with $k = m/\hbar^2 = 1$. This is easily verified for the ground state and the first excited state:

$$\begin{aligned} \langle |x| \rangle_0 &= \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} |x| e^{-x^2} dx = \frac{1}{\pi^{1/2}}, \\ \langle |x| \rangle_1 &= \frac{2}{\pi^{1/2}} \int_{-\infty}^{\infty} |x| x^2 e^{-x^2} dx = \frac{2}{\pi^{1/2}} \end{aligned} \quad (3.12)$$

which agree with the corresponding results from equation (3.11) for the $n = \eta = 0$ state and the $n = 0, \eta = 1$ state. The result in equation (3.11) is interesting, and it leads to the first-order change in the energy as in equation (3.10) and equation (3.11) with bifurcation of the oscillator potential. Indeed we can generalize the results for the case of usual units,

$$\langle \psi ||x| |\psi \rangle = \frac{2}{\pi} \left(2n + \frac{1}{2}\eta + \frac{1}{2}\right) \frac{\Gamma(n + 1/2)}{n!} \left(\frac{\hbar^2}{mk}\right)^{1/4}. \quad (3.13)$$

3.3. Energies for large values of a

For large values of a , we note that equation (2.15) implies that $p \rightarrow 0$, and

$$pa - \eta \frac{\pi}{2} \rightarrow \left(n + \frac{1}{2}\right) \pi \quad \text{for } a \rightarrow \infty. \quad (3.14)$$

Therefore we take

$$(2E)^{1/2} a = \left(n + \frac{\eta}{2} + \frac{1}{2}\right) \pi - \delta, \quad (3.15)$$

and substitute it into equation (2.15) to obtain

$$\begin{aligned} \frac{(2E)^{1/2}}{\delta} &= 2 \frac{\Gamma(3/4)}{\Gamma(1/4)}, \\ \Rightarrow \delta &= \frac{\Gamma(1/4)}{2\Gamma(3/4)} \frac{\left(n + \frac{\eta}{2} + \frac{1}{2}\right) \pi}{a} \quad \text{for } a \rightarrow \infty. \end{aligned} \quad (3.16)$$

Therefore, we take

$$E = \frac{1}{2a^2} \left[\left(n + \frac{\eta}{2} + \frac{1}{2}\right) \pi - \delta \right]^2 \quad (3.17)$$

which leads to

$$E = \frac{\pi^2}{2a^2} \left(n + \frac{\eta}{2} + \frac{1}{2}\right)^2 \left[1 - \frac{\Gamma(1/4)}{\Gamma(3/4)} \frac{1}{a} \right], \quad \text{for } a \rightarrow \infty, \quad (3.18)$$

correct to the first two terms in powers of $1/a$. This result will be useful in developing simple expressions for the energy.

3.4. WKB approximation

The WKB energies are deduced from the relation

$$2 \int_0^{(2E)^{1/2}} \left[2 \left(E_N - \frac{1}{2}y^2 \right) \right]^{1/2} dy + 2 \int_0^a (2E)^{1/2} dx = \left(N + \frac{1}{2} \right) \pi, \quad N = 0, 1, \dots, \tag{3.19}$$

which leads to

$$E_N + \frac{2}{\pi} (2E_N)^{1/2} a - \left(N + \frac{1}{2} \right) = 0, \tag{3.20}$$

$$E_N = \left[\left(\frac{2}{\pi^2} a^2 + N + \frac{1}{2} \right)^{1/2} - \frac{2^{1/2}}{\pi} a \right]^2.$$

For $a \rightarrow 0$, the first two terms are

$$E_N \rightarrow N + \frac{1}{2} - \frac{2}{\pi} (2N + 1)^{1/2} a. \tag{3.21}$$

Taking $N = 2n + \eta$, the first term in this expression agrees with the first term in the exact result in equation (3.7). The second term in equation (3.21) differs from the corresponding term in equation (3.7), but is fairly close for the excited states. The coefficient of the second term in equation (3.21) has values $-0.6366, -1.1027, -1.4235, -1.6843$ for $N = 0, 1, 2, 3$, whereas the corresponding coefficients in the exact relation in equation (3.7) are $-0.5642, -1.1283, -1.4105, -1.6926$. It may also be noted that for $a \rightarrow \infty$, the WKB relation leads to

$$E_N \rightarrow \frac{\pi^2 (N + 1/2)^2}{8a^2}, \quad N = 0, 1, \dots \tag{3.22}$$

This differs from the exact relation in equation (3.18) in that $N + 1/2$ needs to be replaced by $N + 1$. This is essentially related to the end-point corrections in WKB approximations. For large values of N , the WKB result in equation (3.22) tends to the correct expression in equation (3.18), which is expected for the WKB solutions. Overall, the WKB expression for the energy in equation (3.20) may be expected to be quite useful for the excited states, $N = 1, 2, \dots$.

4. Simple expressions for energies

We now develop simple expressions for the energies, using their general properties. Since WKB results would be useful for the excited states, we first consider the ground-state energy, and then the excited state energies using WKB results.

4.1. Ground-state energy

For the ground-state energy, we consider an expression

$$E_0 = \frac{1/2}{1 + c_1 a + c_2 a^2} \tag{4.1}$$

which has the correct value at $a = 0$. Carrying out an expansion in powers of a ,

$$E_0 = \frac{1}{2} (1 - c_1 a + \dots) \tag{4.2}$$

and requiring that it agrees with the corresponding expression from equation (3.7),

$$E_0 = \frac{1}{2} - \frac{1}{\pi} \Gamma(1/2) a, \tag{4.3}$$

one obtains

$$c_1 = \frac{2}{\pi^{1/2}}. \quad (4.4)$$

For large values of a , equation (4.1) for E_0 leads to

$$E_0 \rightarrow \frac{1}{2c_2a^2} \left(1 - \frac{c_1}{c_2} \frac{1}{a} + \dots \right). \quad (4.5)$$

The corresponding leading term in the exact result in equation (3.18) implies

$$c_2 = \frac{4}{\pi^2}. \quad (4.6)$$

Therefore, for the ground-state energy we take

$$E_0 = \frac{1/2}{1 + 2a/\pi^{1/2} + 4a^2/\pi^2}. \quad (4.7)$$

The predictions of this expression for the ground-state energy for a wide range of a are given in table 1, and are found to be in agreement with the exact energies within an accuracy of 1.72% over the entire range of separation parameter a . It may also be noted that with c_1 and c_2 in equations (4.4) and (4.6), we have

$$c_1/c_2 = 2.784 \quad (4.8)$$

for the ratio of coefficients for large a . The corresponding ratio for the exact result in equation (3.18) is

$$\frac{\Gamma(1/4)}{\Gamma(3/4)} \approx 2.959. \quad (4.9)$$

The closeness of the two values indicates the usefulness of the expression in equation (4.7) over the entire domain.

4.2. Excited state energies

For the excited states, we consider an expression

$$E_N = \left[\left(\frac{2a^2}{\pi^2} + N + \frac{1}{2} + \frac{1}{2} e^{-c_N/a} \right)^{1/2} - \frac{2^{1/2}}{\pi} a \right]^2, \quad N = 1, 2, \dots \quad (4.10)$$

This is a small modification of the WKB expression in equation (3.20) in that it has an extra term $(1/2) e^{-c_N/a}$. This does not affect the small a expansion since one has the same expansion for small a as in equation (3.21) where the first term is exact and the second term was found to be quite accurate for $N = 1, 2, \dots$. Now, for large a , we have from equation (4.10),

$$\begin{aligned} E_N &\rightarrow \left[\frac{\pi}{8^{1/2}a} \left(N + 1 - \frac{c_N}{2a} \right) \right]^2, \\ &\rightarrow \frac{\pi^2(N+1)^2}{8a^2} \left[1 - \frac{c_N}{(N+1)a} \right], \quad \text{for } a \rightarrow \infty. \end{aligned} \quad (4.11)$$

Comparing this with the exact relation in equation (3.18), one has

$$c_N = (N+1) \frac{\Gamma(1/4)}{\Gamma(3/4)} = 2.9587(N+1). \quad (4.12)$$

Therefore, we take for the excited state energies,

$$E_N = \left[\left(\frac{2a^2}{\pi^2} + N + \frac{1}{2} + \frac{1}{2} e^{-2.9587(N+1)/a} \right)^{1/2} - \frac{2^{1/2}}{\pi} a \right]^2, \quad N = 1, 2, \dots \quad (4.13)$$

This expression has an accurate representation of the two leading terms for small a , and of the two leading terms for large a . The predicted values of the energies for $N = 1, \dots, 5$, for a wide range of values of a , are given in table 1. The predicted results are in agreement with the exact results within an accuracy of 2.0% over the entire range of the separation parameter a , for all the excited states.

Apart from the practical utility, the simple expression in equation (4.13) provides a very useful physical insight. Going back to the original potential in equation (2.1) and using the scaling relation in equation (2.5), one observes that for

$$N + \frac{1}{2} \gg \frac{2a^2}{\pi^2} \left(\frac{mk}{\hbar^2} \right)^{1/2} \quad (4.14)$$

the energies tend to the simple harmonic oscillator energy levels. This implies that for small a or small k , the effect of bifurcation is small. For

$$N + \frac{1}{2} \ll \frac{2a^2}{\pi^2} \left(\frac{mk}{\hbar^2} \right)^{1/2}, \quad (4.15)$$

the energies tend to the energies for a particle in a box. This implies that for large a or for large k , the bifurcation effect is dominant. In a sense the bifurcation effect is determined by the magnitude of the parameter

$$b = a \left(\frac{mk}{\hbar^2} \right)^{1/4} \frac{1}{2(N + 1/2)^{1/2}}. \quad (4.16)$$

The effect is small for small values of b , and increases with increasing values of b .

4.3. Extension to three dimensions

Our considerations can be extended to the three-dimensional case,

$$V(r) = \frac{1}{2}(r - a)^2\theta(r - a), \quad (4.17)$$

with θ being the Heaviside step function. In particular, we consider the $l = 0$ eigenstates for which the radial equation reduces to

$$-\frac{1}{2} \frac{d^2}{dr^2}(rR_0) + \frac{1}{2}(r - a)^2\theta(r - a)(rR_0) = E(rR_0) \quad (4.18)$$

which is similar to equation (2.1) for the one-dimensional case except that rR_0 vanishes at $r = 0$. We then invoke interdimensional degeneracy [8] that the $l = 0$ solutions for the three-dimensional case correspond to the odd solutions in one dimension. Therefore using equation (4.13), one can write the energies for the $l = 0$ states in three dimensions as

$$E_n^{(0)} = \left[\left(\frac{2a^2}{\pi^2} + 2n + \frac{3}{2} + \frac{1}{2} e^{-2.9587(2n+2)/a} \right)^{1/2} - \frac{2^{1/2}}{\pi} a \right]^2, \quad n = 0, 1, \dots, \quad (4.19)$$

for $m/\hbar^2 = 1$, $k = 1$, and use the scaling relation in equation (2.5) to obtain the solutions for the general case. This is a nice application of interdimensional degeneracy. Another relation which follows from the interdimensional degeneracy is that equation (3.13) with $\eta = 1$ leads to

$$\langle \psi | r | \psi \rangle = \frac{2}{\pi} (2n + 1) \frac{\Gamma(n + 1/2)}{n!} \left(\frac{\hbar^2}{mk} \right)^{1/4}, \quad n = 0, 1, \dots \quad (4.20)$$

for the $l = 0$ states in three dimensions.

5. Summary

We have considered the energy eigenfunctions and eigenvalues for the bifurcated oscillator potential given in equations (1.1) and (2.1). The scaling property as in equation (2.5) implies that we can consider the system in terms of a single separation parameter a . With appropriate wavefunctions in different regions, the continuity conditions lead to a simple relation as in equation (2.15), which allows us to calculate the energies. The relation also allows us to analyse the energies in the small a and large a regions. This leads to some nice results. In particular, combining the small a behaviour with the Feynman–Hellmann theorem leads to general expressions for the average value of $|x|$ as in equation (3.13), for the energy eigenstates. They also allow us to obtain simple expressions for the ground state and excited state energies, for all values of the separation parameter a . These results are extended to the $l = 0$ states of the corresponding potential in three dimensions. Apart from the theoretical interest, the system would be of practical interest for particles in a well with oscillator potential at the edges.

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